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Second syzygy of determinantal ideals generated by minors of generic symmetric matrices

Mitsuyasu Hashimoto[1,*]

Department of Mathematics, School of Science, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan

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Abstract

We prove a characteristic-free plethysm formula of the complex $SS_2\varphi$ for a map of finite free modules φ. We also study a family of subcomplexes of a Schur complex, the *v-Schur complexes*. Using these machineries from characteristic-free representation theory of general linear group, we give an example of determinantal ideal of generic symmetric matrix whose second Betti number depends on the characteristic of the base field. We also give a short proof of Kurano's first syzygy theorem.

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0. Introduction

Let R be a commutative ring with 1, and n a positive integer. Consider a polynomial ring $S = R[x_{ij}]_{1 \le i \le j \le n}$ over R in $n(n+1)/2$ variables. Then we can form a *generic symmetric matrix* $(x_{ij})_{1 \le i,j \le n}$, where $x_{ji} = x_{ij}$ for $i \le j$. Let t be a positive integer. We denote the ideal of S generated by all t-minors of (x_{ij}) by I_t, and call it the *determinantal ideal* of the generic symmetric matrix (x_{ij}). With letting each variable x_{ij} of degree one, the polynomial ring S is graded, and I_t is homogeneous.

As well as other homological properties, there has been much interest in finding graded minimal free resolution of S/I_t as an S-module. Kutz [19] proved that the ideal I_t is perfect of codimension $(n-t+1)(n-t+2)/2$ (i.e., grade $I_t = \mathrm{pd}_S\, S/I_t = (n-1+1)(n-t+2)/2$). It follows that if R is Cohen–Macaulay, then so is S/I_t. He also proved that if R is a domain, then so is S/I_t. It follows that S/I_t is a free R-module (this is true for $R = \mathbb{Z}$, because each homogeneous component of S/I_t is a finitely generated torsion free $\mathbb{Z}$-module, and the general case follows by base change).

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[1] Current address: Nagoya University College of Medical Technology, Higashi-ku, Nagoya 461 Japan.
* E-mail: hasimoto@math.nagoya-u.ac.jp.

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Goto [8] proved that if R is a Krull domain with the class group C, then so is S/I_t with the class group $C \oplus \mathbb{Z}/2\mathbb{Z}$. Goto [9] also proved that S/I_t is Gorenstein if and only if $n-t$ is even and R is Gorenstein.

In [16], Józefiak et al. determined $\mathrm{Tor}_i^S(S/I_t, S/S_+)$ completely, provided R is a field of characteristic zero, where $S_+ = I_1$ is the ideal of S generated by all variables x_{ij}. Note that the Betti number $\beta_i^R = \dim_R \mathrm{Tor}_i^S(S/I_t, S/S_+)$ is the rank of the ith term of the minimal free resolution of S/I_t.

As we mentioned above, the projective dimension of S/I_t depends only on $n-t$, and when $n-t$ increases, then so does $\mathrm{pd}_S\, S/I_t$. If $n-t=0$, then I_t is principal, and the resolution of S/I_t is obvious. For the case $n-t=1$, the minimal free resolution of S/I_t over the ring of integers $\mathbb{Z}$ was constructed by Goto and Tachibana [10] and Józefiak [15]. The resolution is t-linear and of length three. Kurano [17] showed that when $n-t=2$, the Betti number β_i^R is independent of R for all i. It follows that there is a graded minimal free resolution of S/I_t over $\mathbb{Z}$ this case [20]. The resolution is almost t-linear, self-dual, and of length six. There is no explicit construction of the resolution in the literature so far, but the method of [3] is applicable.

However, the construction of minimal free resolution "over $\mathbb{Z}$" is not always possible. Using the characteristic-free representation theory of general linear groups, counterexamples on similar question on generic determinantal ideals [11] and Pfaffian ideals [18] were given.

After these counterexamples, J. Andersen proved that the 5th Betti number of S/I_2 depends on the characteristic when $n \geq 7$ [2]. She took advantage of the fact that S/I_2 is a semigroup ring. She also proved that the 3rd Betti number of S/I_2 is independent of the characteristic of R for any n.

In this article, we make a representation theoretical approach to the resolution problem of S/I_t for arbitrary t. It seems to be impossible to apply semigroup ring approach to the case $t > 2$. Our interest is mainly concentrated into the lower syzygies, and we more or less mimic the method used in [13], which was effective to study the first syzygies of Pfaffian ideals. The basic idea is the use of plethysm formula in complex version.

As a result, we obtained a first example of S/I_t whose 3rd Betti number depends on characteristic (Theorem 7.1). It is also another example of S/I_t without minimal free resolution over $\mathbb{Z}$. It would be interesting to try to compute the increase of the 3rd Betti number at the smallest example – 3-minors of 11×11 matrix. The 3rd Betti number of this ring at characteristic zero is 3 100 383.

We also obtained a new proof of Kurano's first syzygy theorem (Theorem 6.1), which states that the first syzygy module of I_t is generated by degree $t+1$ elements. Note that it follows that β_2^R is independent of R.

In Section 1, we review some basic facts from characteristic-free representation theory. In Section 2, we prove characteristic-free plethysm formula in chain complex version. This generalizes the characteristic-free plethysm formula of SS_2F [17, 4] and $D\bigwedge^2 G$ [4] for finite free R-modules F and G. In Section 3, we introduce the notion of ν-Schur complex, which is a free subcomplex of a Schur complex, and is a generalization of t-Schur complex in [12, 13]. Using plethysm formula, we prove that there is a spectral sequence whose E^1-term is a homology group of certain ν-Schur complex of the identity map, and converges to $[\mathrm{Tor}_i^S(S/I_t, S/S_+)]_j$. Sections 4 and 5 are devoted to study homology groups of ν-Schur complexes of the identity map. The contents of these sections are (more or less straightforward) generalizations of the theory of t-Schur complexes. The goal of these sections is a vanishing theorem Corollary 5.2.

Utilizing the spectral sequence established in Section 3 and Corollary 5.2, we study lower syzygies of S/I_t in Sections 6 and 7. In Section 6, we give a new proof of Kurano's first syzygy theorem, which states that the first syzygy module of I_t is generated by degree $t+1$ elements. In Section 7, we give an example of I_t whose second Betti number depends on characteristic.

1. Preliminaries

Throughout this article, R is a commutative ring with 1. The symbol $\otimes$ means the tensor product $\otimes_R$ over R. We denote the set of positive integers, non-negative integers, integers and rational numbers by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$, respectively. For a prime number p, we denote the prime field of characteristic p by $\mathbb{F}_p$. The symbol $\mathbb{F}_0$ stands for the field of rational numbers $\mathbb{Q}$. For a set X, the cardinality of X is denoted by $\# X$. For a positive integer n, the nth symmetric group is denoted by $\mathfrak{S}_n$.

In this article, a "poset" stands for an ordered set (*p*artially *o*rdered *set*). For a poset P and its subset I, we say that I is a *poset ideal* of P if $x \in P$, $y \in I$ and $x \leq y$ together imply $x \in I$.

Let F be a finite free R-module and $i \geq 0$. We denote by $S_i F$ (resp. $\bigwedge^i F$, $D_i F$) the ith symmetric power (resp. exterior power, divided power) of F. The symmetric algebra (resp. exterior algebra, divided power algebra) of F is denoted by SF (resp. $\bigwedge F$, DF). For a map of finite free R-modules $\varphi: G \to F$, the ith symmetric power (resp. exterior power) of φ is denoted by $S_i\varphi$ (resp. $\bigwedge^i \varphi$). The symmetric (resp. exterior) algebra of φ is denoted by $S\varphi$ (resp. $\bigwedge \varphi$). For a finite free R-complex

$$\alpha: 0 \to G \xrightarrow{\psi} F \xrightarrow{\varphi} E \to 0$$

of length at most two, we denote the ith symmetric power (resp. the symmetric algebra) of α by $S_i\alpha$ (resp. $S\alpha$). For these multilinear objects, we refer the reader to [1, 14].

In this paper, these multilinear objects will be considered in the category of bigraded R-modules G_R^2 or in the category of graded R-complexes $\mathscr{C}$ as in [12–14]. In particular, the definition of "bialgebras" is slightly different from the usual one (only by sign, in some sense). See [14, Chapter I] for details. For an object $C \in \mathscr{C}$ and $n \in \mathbb{Z}$, we define $C[n]$ by $C[n]_{i,j} := C_{i,j+n}$ and $\partial_j^{C[n]} := (-1)^n \partial_{j+n}^C$.

The algebras SF, $\bigwedge F$, DF, $S\varphi$, $\bigwedge\varphi$ and $S\alpha$ have bialgebra structures. See [14, Chapter I] for details.

Let B be an R-algebra. The multiplication map $B \otimes B \to B$ is denoted by m_B. If there is no danger of confusion, it is simply denoted by m.

For an R-coalgebra A, we denote the coproduct of A by Δ_A or simply by Δ. If A is a bigraded R-bialgebra, the composite map

$$A \otimes A \xrightarrow{\Delta_A \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes m} A \otimes A$$

is denoted by $\square_A$, or simply by $\square$, and called the *box map* of A.

A row-sequence is an infinite sequence of non-negative integers, say $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, such that $\lambda_i \neq 0$ for only finitely many i. If $\lambda_i = 0$ for $i > r$, then we may write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$. Addition (or subtraction) of two row-sequences, and scalar multiple of a row-sequence is defined as those of vectors. We denote the row-sequence $(\delta_{i1}, \delta_{i2}, \dots)$ by ε_i for $i \geq 1$, where δ is the Kronecker's delta. We define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \geq 1$. Note that α_i is not a row-sequence. A *partition* is a weakly decreasing row-sequence by definition.

For two row-sequences λ and μ, we say that $\lambda \supset \mu$ when $\lambda_i \supset \mu_i$ for $i \geq 1$. We say that $\lambda \succeq \mu$ when $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j$ for $i \geq 1$. We say that $\lambda > \mu$ when there exists some $i \geq 1$ such that $\lambda_j = \mu_j$ for all $j < i$ and that $\lambda_i > \mu_i$. It is easy to see that we have

$$\lambda \supset \mu \quad \Rightarrow \quad \lambda \succeq \mu \quad \Rightarrow \quad \lambda \geq \mu.$$

Note that the order $\geq$ is a total order on the set of row-sequences.

A *relative row-sequence* λ/μ is a pair of row-sequences (λ, μ) such that $\lambda \supset \mu$. We may write simply λ instead of $\lambda/0$, where $0 = (0, 0, \dots)$ is the zero partition. The diagram $\Delta_{\lambda/\mu}$ of λ/μ is the subset

$$\{(i,j) \in \mathbb{N}^2 \mid \mu_i < j \leq \lambda_i\}$$

of $\mathbb{N}^2$ by definition. If both λ and μ are partitions, then λ/μ is called a *skew partition.* The *degree* of a relative row-sequence λ/μ, denoted by $|\lambda/\mu|$, is $\# \Delta_{\lambda/\mu}$ by definition. The *length* of λ/μ, denoted by $l(\lambda/\mu)$ is

$$\max(\{0\} \cup \{i \in \mathbb{N} \mid \lambda_i > \mu_i\})$$

by definition.

Let $\varphi : G \to F$ be a map of finite free R-modules, and λ/μ a relative row-sequence. We define

$$\bigwedge_{\lambda/\mu} \varphi := \bigwedge^{\lambda_1 - \mu_1} \varphi \otimes \bigwedge^{\lambda_2 - \mu_2} \varphi \otimes \cdots \otimes \bigwedge^{\lambda_r - \mu_r} \varphi,$$

where $r \geq l(\lambda/\mu)$.

We denote by $L_{\lambda/\mu} F$ (resp. $L_{\lambda/\mu} \varphi$) the Schur module of F (resp. the Schur complex of φ) with respect to λ/μ. For $t \geq 0$, the t-Schur complex of φ with respect to λ/μ is denoted by $L_{t,\, \lambda/\mu} \varphi$. For the results, notation and terminology related to these objects (such as standardness of tableaux, standard basis theorem) and other related (unexplained) notation and terminology, we refer the reader to [1, 14, 12].

However, *we use one different notation*. The complex $\bigwedge_{t,1,\lambda/\mu}\varphi$ in [12] is denoted simply by $\bigwedge_{t,\lambda/\mu}\varphi$ in this paper, and $\bigwedge_{t,\lambda/\mu}\varphi = \sum_i \bigwedge_{t,i,\lambda/\mu}\varphi$ in [12] will never be used in this paper. This notation is the same as that in [13].

Let $\psi : G' \to F'$ be a map of finite free R-modules, too. In [14, Chapter III], a coalgebra homomorphism

$$\theta(\varphi,\psi): \bigwedge \varphi \otimes \bigwedge \psi \to S(\varphi \otimes \psi)$$

is defined. The map θ is uniquely determined by the property:

> The map $\theta(\varphi,\psi)$ depends only on F, G, F' and G', and is a universal natural transformation on F, G, F' and G'. It is a homomorphism of coalgebras in $\mathscr{C}$, and $\theta|_{\bigwedge^1\varphi\otimes\bigwedge^1\psi} : \bigwedge^1\varphi\otimes\bigwedge^1\psi = \varphi\otimes\psi \to \varphi\otimes\psi = S_1(\varphi\otimes\psi)$ is the identify. (1.1)

We call θ the *generalized determinant map*.

We recall that the restriction of θ to $\bigwedge^i F \otimes \bigwedge^j G$ is zero when $i \neq j$, and it is given by

$$\theta(f_1 \wedge \cdots \wedge f_i \otimes g_1 \wedge \cdots \wedge g_i) = (-1)^{i(i-1)/2}\det(f_\alpha \otimes g_\beta)_{1\le\alpha,\beta\le i} \tag{1.2}$$

for $f_1,\ldots,f_i \in F$ and $g_1,\ldots,g_i \in G$, when $i = j$.

2. Generalized plethysm formula

In this section, we prove a generalization of the plethysm formula of SS_2F for a finite free R-module F [17] to the chain complex version. The generalized version is also a generalization of the plethysm formula of $D\bigwedge^2 F$ [4]. The proof is similar to that of the generalized plethysm for pfaffians [13].

Let $\varphi : G \to F$ be a map of finite free R-modules. Then, the complex $S_2\varphi$ is of length (at most) two, and is of the form

$$0 \to \bigwedge^2 G \to F \otimes G \to S_2 F \to 0.$$

So we obtain the symmetric algebra $SS_2\varphi$ of $S_2\varphi$. We denote the composite map

$$\bigwedge\varphi\otimes\bigwedge\varphi \xrightarrow{\theta} S(\varphi\otimes\varphi) \xrightarrow{Sm} S(S_2\varphi)$$

by $\bar{\theta}$, where $m : \varphi\otimes\varphi \to S_2\varphi$ is the multiplication map (note that $S(?)$ is a functor). Since θ and Sm are coalgebra maps, so is $\bar{\theta}$. The restriction of $\bar{\theta}$ to $\bigwedge^r\varphi\otimes\bigwedge^r\varphi$ is denoted by $\bar{\theta}_r$.

Lemma 2.1. *Let $0 < k \le r$. Then, we have the composite map*

$$\bigwedge^{r+k}\varphi\otimes\bigwedge^{r-k}\varphi \xrightarrow{\square_{(r,r)}^{(r+k,r-k)}} \bigwedge^r\varphi\otimes\bigwedge^r\varphi \xrightarrow{\bar{\theta}_r} S_r(S_2\varphi)$$

is zero, where $\square_{(r,r)}^{(r+k,r-k)}$ is the box map (see [12, *Definition* 1.1.2] *for definition).*

Proof. As the maps in consideration are universal (i.e., compatible with the base change) and defined over an integral domain of characteristic zero, we may assume that $S(S_2\varphi)$ is cogenerated by $S_2\varphi$ (see [13, Section 2]).

First, we have that two coalgebra maps

$$\bigwedge \varphi \otimes \bigwedge \varphi \xrightarrow{\bar{\theta}} S(S_2\varphi)$$

and

$$\bar{\theta}' : \bigwedge \varphi \otimes \bigwedge \varphi \xrightarrow{\square} \bigwedge \varphi \otimes \bigwedge \varphi \xrightarrow{\bar{\theta}} S(S_2\varphi)$$

agree, where $\square = (1 \otimes m) \circ (\Delta \otimes 1)$ is the box map. To verify this, we only have to check it at the degree 2 component (with respect to the naive grading) since $S(S_2\varphi)$ is cogenerated by its degree 2 component $S_2\varphi$. In fact, the two maps are zero on $\bigwedge^2 \varphi \otimes \bigwedge^0 \varphi$ and on $\bigwedge^0 \varphi \otimes \bigwedge^2 \varphi$, while they are the multiplication of $S\varphi$ over $\bigwedge^1 \varphi \otimes \bigwedge^1 \varphi = \varphi \otimes \varphi$. As the map in the lemma is an appropriate component of the zero map $\bar{\theta}' - \bar{\theta}$, the assertion follows. $\square$

Using a similar argument, we also have

Lemma 2.2. *The composite map*

$$\bigwedge^r \varphi \otimes \bigwedge^r \varphi \xrightarrow{T} \bigwedge^r \varphi \otimes \bigwedge^r \varphi \xrightarrow{\bar{\theta}} S(S_2\varphi)$$

agree with $\bar{\theta}$, where T is an appropriate twisting (for definition, see [14, *p.* 6]).

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a row-sequence (i.e., a sequence of non-negative integers). Then, we denote the row-sequence

$$(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_l, \lambda_l)$$

(obtained by repeating each term of λ twice) by $\hat{\lambda}$.

For a row-sequence λ, we denote the restriction of the composite map

$$\left(\bigwedge \varphi \otimes \bigwedge \varphi\right) \otimes \cdots \otimes \left(\bigwedge \varphi \otimes \bigwedge \varphi\right) \xrightarrow{\bar{\theta} \otimes \cdots \otimes \bar{\theta}} S(S_2\varphi) \otimes \cdots \otimes S(S_2\varphi) \xrightarrow{m} S(S_2\varphi)$$

on

$$\bigwedge_{\hat{\lambda}} \varphi = \bigwedge^{\lambda_1} \varphi \otimes \bigwedge^{\lambda_1} \varphi \otimes \cdots \otimes \bigwedge^{\lambda_l} \varphi \otimes \bigwedge^{\lambda_l} \varphi$$

by $\bar{\theta}_\lambda$.

Definition 2.3. Let $r \geq 0$. For a partition λ of degree r, we define

$$M_\lambda(\bar{\theta}) := \sum_{\mu \geq \lambda, |\mu| = r} \operatorname{Im} \bar{\theta}_\mu \quad \text{and} \quad \dot{M}_\lambda(\bar{\theta}) := \sum_{\mu > \lambda, |\mu| = r} \operatorname{Im} \bar{\theta}_\mu.$$

Lemma 2.4. $M_{(1^r)}(\bar{\theta}) = S_r(S_2\varphi)$.

Proof. (*By induction on r*). The assertion is obviously true when $r \le 1$. Consider the case $r \ge 2$. As $S(S_2\varphi) = S(S_2F) \otimes \bigwedge(F \otimes G) \otimes D(\bigwedge^2 G)$ and the algebras $S(S_2\varphi)$ and $\bigwedge(F \otimes G)$ are generated by their degree one components, we may assume that $F = 0$ by induction assumption.

Thus, what we want to prove is $M_{(1^r)}(\bar{\theta}) = D_r(\bigwedge^2 G)$ when $F = 0$. But we know that $M_{(1^r)}(\theta) = D_r(G \otimes G)$ (see [14, Lemma III.2.5]), where

$$M_\lambda(\theta) := \sum_{\mu \ge \lambda, |\mu| = r} \operatorname{Im} \theta_\mu$$

for a partition λ of degree r. As $m: G \otimes G \to \bigwedge^2 G$ is a split epimorphism, so is $D_r m: D_r(G \otimes G) \to D_r(\bigwedge^2 G)$, and the assertion follows. □

Lemma 2.5. *Let $r \ge 0$. Then, we have*

$$\operatorname{rank}_R(S_r(S_2\varphi)) = \sum_\lambda \operatorname{rank}_R L_{\hat{\lambda}}\varphi$$

where the sum is taken over all partitions of degree r.

Proof. Similar to [13, Lemma 3.8], and we omit it. □

Theorem 2.6. *Let $r \ge 0$ and λ be a partition of degree r. Then, there exists a unique isomorphism*

$$\gamma_\lambda : L_{\hat{\lambda}}\varphi \to M_\lambda(\bar{\theta})/\dot{M}_\lambda(\bar{\theta})$$

such that the diagram

$$\begin{array}{ccc} \bigwedge_{\hat{\lambda}}\varphi & \xrightarrow{\bar{\theta}_\lambda} & M_\lambda(\bar{\theta}) \\ {\scriptstyle d_{\hat{\lambda}}}\downarrow & & \downarrow \\ L_{\hat{\lambda}}\varphi & \xrightarrow{\gamma_\lambda} & M_\lambda(\bar{\theta})/\dot{M}_\lambda(\bar{\theta}) \end{array}$$

is commutative. So $S_r(S_2\varphi)$ is isomorphic to $\bigoplus_{|\lambda| = r} L_{\hat{\lambda}}\varphi$ up to filtration.

Proof. To see that γ_λ is induced, it suffices to show that the image of the composite map

$$\bigwedge_\mu \varphi \xrightarrow{\square_{\hat{\lambda}}^{\mu}} \bigwedge_{\hat{\lambda}}\varphi \xrightarrow{\bar{\theta}_\lambda} M_\lambda(\bar{\theta})$$

is contained in $\dot{M}_\lambda(\bar{\theta})$ for any $\mu \in S_\square(\lambda)$ (see [12, p. 462]). We may write $\mu = \hat{\lambda} + k\alpha_1$ $(k, l > 0)$.

When l is odd, then we may assume that $l = 1$, and this case is reduced to Lemma 2.1. When l is even, then we may assume that $l = 2$, and $\lambda = (\lambda_1, \lambda_2)$. Then, the composite map

$$\bigwedge_\mu \varphi = \bigwedge^{\lambda_1}\varphi \otimes \bigwedge^{\lambda_1+k}\varphi \otimes \bigwedge^{\lambda_2-k}\varphi \otimes \bigwedge^{\lambda_2}\varphi \xrightarrow{\square_{\tilde\lambda}^{\mu}} \bigwedge_{\tilde\lambda}\varphi \xrightarrow{\bar\theta_\lambda} S(S_2\varphi)$$

agrees with the composite map

$$\bigwedge_\mu \varphi = \bigwedge^{\lambda_1}\varphi \otimes \bigwedge^{\lambda_1+k}\varphi \otimes \bigwedge^{\lambda_2-k}\varphi \otimes \bigwedge^{\lambda_2}\varphi \xrightarrow{1\otimes T} \bigwedge_{\lambda}\varphi \otimes \bigwedge_{\lambda+k\alpha_1}\varphi$$

$$\xrightarrow{1\otimes\square_\lambda} \bigwedge_{\lambda}\varphi \otimes \bigwedge_{\lambda}\varphi \xrightarrow{\theta_\lambda} S(\varphi\otimes\varphi) \xrightarrow{Sm} S(S_2\varphi),$$

by Lemma 2.2. By [14, Proposition III.2.6], its image is contained in $\dot{M}_\lambda(\bar\theta)$, as desired. □

3. ν-Schur complex and determinantal ideal

As in the previous section, $\varphi: G \to F$ is a map of finite free R-modules. For $r \ge k \ge 0$, we denote the truncated subcomplex

$$0 \to \bigwedge^k F \otimes D_{r-k}G \to \bigwedge^{k+1}F \otimes D_{r-k-1}G \to \cdots \to \bigwedge^{r-1}F\otimes G \to \bigwedge^r F \to 0$$

of $\bigwedge^r\varphi$ by $\bigwedge^{k,r}\varphi$. For row-sequences $\mu \subset \gamma \subset \lambda$, we define

$$\bigwedge_{\gamma,\lambda/\mu}\varphi := \bigwedge^{\gamma_1-\mu_1,\lambda_1-\mu_1}\varphi \otimes \bigwedge^{\gamma_2-\mu_2,\lambda_2-\mu_2}\varphi \otimes \bigwedge^{\gamma_3-\mu_3,\lambda_3-\mu_3}\varphi \otimes \cdots$$

so that $\bigwedge_{t,\lambda/\mu}\varphi$ in [13] is $\bigwedge_{\mu+t\varepsilon_1,\lambda/\mu}\varphi$.

Definition 3.1. Let λ/μ be a relative row-sequence, and ν a row-sequence. We denote the subcomplex

$$\sum_\gamma \bigwedge_{\gamma,\lambda/\mu}\varphi \subset \bigwedge_{\lambda/\mu}\varphi$$

by $\bigwedge_{\succeq\nu,\lambda/\mu}\varphi$, where the sum is taken over row-sequences γ such that $\mu\subset\gamma\subset\lambda$ and that $\gamma \succeq \nu$. Assume moreover that λ/μ is a skew-partition. We call the complex $d_{\lambda/\mu}(\bigwedge_{\succeq\nu,\lambda/\mu}\varphi)$ the *ν-Schur complex* of φ with respect to the skew partition λ/μ, and denote it by $L_{\nu,\lambda/\mu}\varphi$.

By definition, $L_{\nu,\lambda/\mu}\varphi = 0$ unless $\nu \preceq \lambda$. Note that the t-Schur complex $L_{t,\lambda/\mu}\varphi$ [12, 13] is nothing but $L_{\mu+t\varepsilon_1,\lambda/\mu}\varphi$. We fix bases $X=\{x_1<\cdots<x_m\}$ of F and $Y=\{y_1>\cdots>y_n\}$ of G, respectively. We let $X<Y$ so that $Z:=X\cup Y$ is a totally ordered set. For a relative row-sequence (resp. skew partition) λ/μ, we denote the set of row-standard (resp. standard) tableaux mod Y of shape λ/μ with values in Z by $\mathrm{Row}_{\lambda/\mu}(Z,Y)$ (resp. $\mathrm{St}_{\lambda/\mu}(Z,Y)$) (see [14, Section I.3]). For $S\in\mathrm{Row}_{\lambda/\mu}(Z,Y)$ and a poset ideal I of Z, there exists a unique row-sequence γ such that $\mu\subset\gamma\subset\lambda$ and that

$S^{-1}(I) = \Delta_{\lambda/\mu}$. We denote this γ by $\gamma(S, I)$. For two tableaux $S, S' \in \mathrm{Row}(Z, Y)$, it holds $S \leq S'$ if and only if $\gamma(S, I) \succeq \gamma(S', I)$ for any poset ideal I of Z (see [12, p. 461]).

For a relative row-sequence λ/μ and a row-sequence $\nu \supset \mu$, we set

$$\mathrm{Row}_{\succeq\nu,\lambda/\mu}(Z, Y) := \{S \in \mathrm{Row}_{\lambda/\mu}(Z, Y) \mid \gamma(S, X) \succeq \nu\}.$$

Note that $S \in \mathrm{Row}_{\lambda/\mu}(Z, Y)$ is contained in $\bigwedge_{\succeq\nu,\lambda/\mu}\varphi$ if and only if $S \in \mathrm{Row}_{\succeq\nu,\lambda/\mu}(Z, Y)$. It is easy to see that $\mathrm{Row}_{\succeq\nu,\lambda/\mu}(Z, Y)$ is a free basis of $\bigwedge_{\succeq\nu,\lambda/\mu}\varphi$.

Lemma 3.2. *Let λ/μ be a skew partition, and ν a row-sequence. The set*

$$\mathrm{St}_{\nu,\lambda/\mu}(Z, Y) := \mathrm{Row}_{\succeq\nu,\lambda/\mu}(Z, Y) \cap \mathrm{St}_{\lambda/\mu}(Z, Y)$$

is a free basis of the ν-Schur complex $L_{\nu,\lambda/\mu}\varphi$. In particular, $L_{\nu,\lambda/\mu}\varphi$ agrees with the sum $\sum_\gamma d_{\lambda/\mu}(\bigwedge_{\gamma,\lambda/\mu}\varphi)$ with sum taken only over all partitions such that $\mu \subset \gamma \subset \lambda$ and that $\gamma \succeq \nu$.

Proof. The first assertion immediately follows from [12, Lemma 1.1.1], as the subset $\mathrm{Row}_{\succeq\nu,\lambda/\mu}(Z, Y)$ is a poset ideal of $\mathrm{Row}_{\lambda/\mu}(Z, Y)$. For any element S in $\mathrm{St}_{\nu,\lambda/\mu}(Z, Y)$, $\gamma(S, X)$ is a partition. The second assertion follows from this. □

Lemma 3.3. *Let λ/μ and ν be as above. Then, one of the following hold.*
1. *$\nu \not\preceq \lambda$. In this case, we have $L_{\nu,\lambda/\mu}\varphi = 0$ for any φ.*
2. *There exists a unique partition ρ such that $\mu \subset \rho \subset \lambda$ and that it holds $\gamma \succeq \nu$ if and only if $\gamma \succeq \rho$ for any partition γ such that $\mu \subset \gamma \subset \lambda$. In this case, we have $L_{\nu,\lambda/\mu}\varphi = L_{\rho,\lambda/\mu}\varphi$* for any φ.

Proof. If there are i and j such that $i < j$, $\mu_i > \nu_i$ and $\mu_j < \nu_j$, then we take such a pair (i, j) such that $j - i$ is small as possible. Then we may replace ν by $\nu' = \nu + \varepsilon_i - \varepsilon_j$ in the sense for any row-sequence $\gamma \supset \mu$, $\gamma \succeq \nu$ if and only if $\gamma \succeq \nu'$. This replacement step cannot be continued infinitely, so we may assume that there is no such a pair (i, j).

Thus, there exists some $r \geq 0$ such that

$$(\mu_1, \ldots, \mu_r) \subset (\nu_1, \ldots, \nu_r)$$

and that

$$(\mu_{r+1}, \mu_{r+2}, \ldots) \supset (\nu_{r+1}, \nu_{r+2}, \ldots).$$

Then we may replace ν by $(\nu_1, \ldots, \nu_r, \mu_{r+1}, \mu_{r+2}, \ldots)$ in the sense above, and we may assume that $\nu \supset \mu$.

If there is i and j such that $i < j$, $\lambda_i > \nu_i$ and $\lambda_i < \nu_j$, then we take such a pair (i, j) such that $j - i$ is as small as possible. Then we may replace ν by $\nu' = \nu + \varepsilon_i - \varepsilon_j$ in the sense for any row-sequence $\gamma \subset \lambda$ it holds $\gamma \succeq \nu$ if and only if $\gamma \succeq \nu'$. Note that the condition $\nu \supset \mu$ is preserved by this replacement.

This replacement cannot be continued infinitely, so we may assume that there exists some $r \geq 0$ such that

$$(\lambda_1, \ldots, \lambda_r) \subset (\nu_1, \ldots, \nu_r)$$

and that

$$(\lambda_{r+1}, \lambda_{r+2}, \ldots) \supset (\nu_{r+1}, \nu_{r+2}, \ldots).$$

Now if $\nu \not\preceq \lambda$, then the condition 1 holds. So we may assume $\nu \not\preceq \lambda$. By the condition above, then it holds that $\mu \subset \nu \subset \lambda$.

If there exists some i such that $\nu_i < \nu_{i+1}$, then we take the minimum i among such, and replace ν by $\nu' = \nu + \alpha_i$ in the sense for any partition γ, $\gamma \succeq \nu$ if and only if $\gamma \succeq \nu'$. Note that the condition $\mu \subset \nu \subset \lambda$ is not violated with this replacement. This replacement step cannot be continued infinitely, and we may assume that there is no such an i, namely, ν is a partition. Now $\rho = \nu$ is the desired partition. The uniqueness is trivial. □

By the lemma above, the assumption ν is a partition such that $\mu \subset \nu \subset \lambda$ is not a serious restriction when we consider $L_{\nu,\lambda/\mu}\varphi$. For a skew partition λ/μ and a row-sequence ν with $\nu \preceq \lambda$, we denote the partition ρ in the lemma by $\rho(\lambda/\mu, \nu)$.

We set $S = S(S_2F)$. Then, the complex $S(S_2\varphi) = S \otimes \bigwedge(F \otimes G) \otimes D(\bigwedge^2 G)$ is a graded S-free complex with letting $S_r(S_2\varphi)$ of degree r. Let I_t be the determinantal ideal (of S) generated by all t-minors of the generic symmetric matrix $(x_i \times x_j)_{1 \leq i, j \leq m}$, where $\times$ is the multiplication in SF (so that each $x_i \times x_j$ belongs to $S_2E \subset S(S_2E)$). In other words, I_t is the ideal of S generated by $\bar{\theta}(\bigwedge_{(t,t)} F)$. The complex $\mathscr{I}^t = \mathscr{I}^t(\varphi) := I_t \otimes_S S(S_2\varphi)$ is an S-graded subcomplex of $S(S_2\varphi)$ in a natural way, whose degree r component is denoted by $\mathscr{I}^{t,r}(\varphi)$ or $\mathscr{I}^{t,r}$.

We are mainly interested in the case $G = F$ and $\varphi = \mathrm{id}_F$.

We define a chain map $\pi : S_2\mathrm{id}_F \to \mathrm{id}_{S_2}F$ as follows:

$$\begin{array}{lccccccccc}
S_2(\mathrm{id}_F): & 0 & \longrightarrow & \bigwedge^2 F & \xrightarrow{\Delta} & F \otimes F & \xrightarrow{m} & S_2F & \longrightarrow & 0 \\
& & & \downarrow 0 & & \downarrow m & & \downarrow 1 & & \\
\mathrm{id}_{S_2F}: & 0 & \longrightarrow & 0 & \longrightarrow & S_2F & \xrightarrow{1} & S_2F & \longrightarrow & 0
\end{array}$$

Lemma 3.4. *The map*

$$1 \otimes S\pi : \mathscr{I}^t(\mathrm{id}_F) = I_t \otimes_S S(S_2\,\mathrm{id}_F) \to I_t \otimes_S S(\mathrm{id}_{S_2F})$$

is a quasi-isomorphism of GL(F)*-equivariant graded* S*-complex.*

Proof. Similar to the proof of [13, Lemma 7.1], and is left to the reader. □

By the lemma, we have an isomorphism

$$H_i(\mathscr{I}^{t,j}(\mathrm{id}_F)) \cong [\mathrm{Tor}^S_{i+1}(S/I_t, S/S_+)]_j \tag{3.1}$$

for $i, j \geq 0$, since $S(\mathrm{id}_{S_2 F})$ is nothing but the Koszul complex $K_*(x_i \times x_j; S)$, which is a free resolution of $S/S_+ = R$. Here $S_+ = (x_i \times x_j \mid 1 \leq i, j \leq m) \cdot S$, and $[\,]_j$ denotes the degree j component of a graded S-module.

Now we return to the case of general $\varphi : G \to F$.

Definition 3.5. Let $r \geq 0$ and $t \geq 1$. For a partition λ of degree r, we define

$$M_{t,\lambda}(\bar{\theta}) := \sum_{\mu \geq \lambda, |\mu| = r} \bar{\theta}_{\tilde{\mu}}\left(\bigwedge\nolimits_{\geq \nu(t,\mu), \tilde{\mu}} \varphi\right)$$

and

$$\dot{M}_{t,\lambda}(\bar{\theta}) := \sum_{\mu > \lambda, |\mu| = r} \bar{\theta}_{\tilde{\mu}}\left(\bigwedge\nolimits_{\geq \nu(t,\mu), \tilde{\mu}} \varphi\right),$$

where $\nu(t, \mu)$ is the smallest two-rowed partition of degree $\mu_1 + t$ (in other words, the partition $(\mu_1 + t - [(\mu_1 + t)/2], [(\mu_1 + 1)/2])$).

It is clear that $\dot{M}_{t,\lambda}(\bar{\theta}) \subset M_{t,\lambda}(\bar{\theta})$. By [14, Lemma IV.1.7], we have $M_{t,\lambda} \subset \mathscr{I}^{t,r}$ for any λ. By definition, $\mathscr{I}^{t,r}$ is the image of multiplication map $\mathscr{I}^{t,t} \otimes S_{r-t}(S_2\varphi) - S_r(S_2\varphi)$. It is easy to see that $\mathscr{I}^{t,t} = I_{t,t} = M_{t,(t)}(\bar{\theta})$, where $I_{t,t}$ is the degree t component of I_t. Hence, we have $M_{t,(t,1^{r-t})} = \mathscr{I}^{t,r}$ so that

$$\mathscr{I}^{t,r} = M_{t,(t,1^{r-t})} \supset \cdots \supset M_{t,(r)} \supset 0$$

is a filtration of $\mathscr{I}^{t,r}$.

Lemma 3.6. *Let $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2)$ and $\lambda = (\lambda_1, \lambda_2)$ be two-rowed partitions such that $\mu \subset \nu \subset \lambda$. If $\nu_1 \geq \lambda_2$, then we have*

$$\operatorname{Im} \square_{\lambda/\mu} \cap \bigwedge\nolimits_{\geq \nu, \lambda/\mu} \varphi = \sum_{k > \mu_1 - \mu_2} \square_{\lambda/\mu}\left(\bigwedge\nolimits_{\geq \nu - \mu + k\alpha_1, \lambda - \mu + k\alpha_1} \varphi\right).$$

Proof. We set $t = \nu_1 - \mu_1$. By [13, Lemma 7.4], we have

$$\operatorname{Im} \square_{\lambda/\mu} \cap \bigwedge\nolimits_{t, \lambda/\mu} \varphi = \sum_{k > \mu_1 - \mu_2} \square_{\lambda/\mu}\left(\bigwedge\nolimits_{t+k, \lambda - \mu + k \cdot \alpha_1} \varphi\right).$$

Truncating the complexes of both-hand sides at degree $|\nu/\mu|$, we obtain the desired equality. $\square$

Lemma 3.7. *If $|\lambda| \leq 2t$, then we have*

$$\dot{M}_{t,\lambda}(\bar{\theta}) = \dot{M}_{\lambda}(\bar{\theta}) \cap M_{t,\lambda}(\bar{\theta}).$$

In particular, we have an isomorphism

$$\gamma_{t,\lambda} : L_{\nu(t,\lambda)}\varphi \to M_{t,\lambda}(\bar{\theta})/\dot{M}_{t,\lambda}(\bar{\theta})$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \bigwedge_{\succeq \nu(t,\lambda),\tilde{\lambda}}\varphi & \xrightarrow{\bar{\theta}_\lambda} & M_{t,\lambda} \\ d_{\tilde{\lambda}} \big\downarrow & & \big\downarrow \\ L_{\nu(t,\lambda),\tilde{\lambda}}\varphi & \xrightarrow{\gamma_{t,\lambda}} & M_{t,\lambda}/\dot{M}_{t,\lambda} \end{array}$$

In particular, if $r \leq 2t$, then $\mathscr{F}^{t,r}$ is isomorphic to $\bigoplus_{|\lambda|=r} L_{\nu(t,\lambda),\tilde{\lambda}}\varphi$ up to filtration.

Proof. Note that $\gamma_\lambda : L_{\tilde{\lambda}}\varphi \to M_\lambda/\dot{M}_\lambda$ (see Theorem 2.6) maps

$$L_{\nu(t,\lambda),\tilde{\lambda}}\varphi = d_{\tilde{\lambda}}(\textstyle\bigwedge_{\succeq \nu(t,\lambda),\tilde{\lambda}}\varphi)$$

isomorphically onto

$$(\bar{\theta}_\lambda(\textstyle\bigwedge_{\succeq \nu(t,\lambda),\tilde{\lambda}}\varphi) + \dot{M}_\lambda)/\dot{M}_\lambda = (M_{t,\lambda} + \dot{M}_\lambda)/\dot{M}_\lambda.$$

If $\dot{M}_{t,\lambda} = \dot{M}_\lambda \cap M_{t,\lambda}$, then we define $\gamma_{t,\lambda}$ to be the composite isomorphism

$$L_{\nu(t,\lambda),\tilde{\lambda}}\varphi \xrightarrow{\gamma_\lambda} (M_{t,\lambda} + \dot{M}_\lambda)/\dot{M}_\lambda \cong M_{t,\lambda}/(\dot{M}_\lambda \cap M_{t,\lambda}) = M_{t,\lambda}/\dot{M}_{t,\lambda},$$

and the rest of the assertions follow.

So it suffices to prove $\dot{M}_{t,\lambda} = \dot{M}_\lambda \cap M_{t,\lambda}$. The direction $\subset$ is clear. We prove the direction $\supset$. By Theorem 2.6, it suffices to prove

$$\bar{\theta}_\lambda\left(\operatorname{Ker} d_{\tilde{\lambda}} \cap \textstyle\bigwedge_{\succeq \nu(t,\lambda),\tilde{\lambda}}\varphi\right) \subset \dot{M}_{t,\lambda}.$$

By [13, Lemma 7.5], the left-hand side is equal to $\sum_{i=1}^{l(\tilde{\lambda})} E(i)$, where

$$E(i) = \bar{\theta}_\lambda\left(\left(\sum_{k>0} \operatorname{Im} \square_{\tilde{\lambda}}^{\tilde{\lambda}+k\alpha_i}\right) \cap \textstyle\bigwedge_{\succeq \nu(t,\lambda),\tilde{\lambda}}\varphi\right).$$

What we want to prove is that $E(i) \subset \dot{M}_{t,\lambda}$ for $i \geq 1$. By Lemma 2.1, $E(i) = 0$ for i odd. So we may assume that i is even. Clearly, we may assume that $\lambda_1 \geq t$. In this case, we have $\lambda_1 \geq \lambda_2 + \lambda_3 + \cdots$. So the case $i \geq 4$ is almost trivial. So we consider the case $i = 2$, and in this case, we may assume that $l(\lambda) = 2$. Note that we have $\lambda_2 \leq t$. By [13, Lemma 7.4], we have

$$E(2) \subset \sum_{\nu(t,\lambda) \preceq \gamma \subset (\lambda_1,\lambda_1)} \sum_{k>0} Sm\left(\theta_\lambda\left(\textstyle\bigwedge_{\gamma_1,\lambda}\varphi \otimes \square_\lambda\left(\bigwedge_{\gamma_2+k,\lambda+k\alpha_1}\varphi\right)\right)\right), \tag{3.2}$$

where $Sm : S(\varphi \otimes \varphi) \to S(S_2\varphi)$ is the map induced by the multiplication $m : \varphi \otimes \varphi \to S_2\varphi$. By [14, Proposition III.2.6], we have

$$\theta_\lambda\left(\textstyle\bigwedge_{\gamma_1,\lambda}\varphi \otimes \square_\lambda\left(\bigwedge_{\gamma_2+k,\lambda+k\alpha_1}\varphi\right)\right) \subset \theta_\lambda\left(\bigwedge_{\gamma_1,\lambda+k\alpha_1}\varphi \otimes \bigwedge_{\gamma_2+k,\lambda+k\alpha_1}\varphi\right). \tag{3.3}$$

For any γ such that $\nu(t,\lambda)\preceq\gamma\subset(\lambda_1,\lambda_1)$ and $k>0$, we have $\gamma_1+(\gamma_2+k)\geq t+(\lambda_1+k)$. So we have $(\gamma_1,\gamma_2+k)\succeq\nu(t,\lambda+k\alpha_1)$ or $(\gamma_2+k,\gamma_1)\succeq\nu(t,\lambda+k\alpha_1)$. This shows that

$$Sm\left(\theta_\lambda\left(\bigwedge\nolimits_{\gamma_1,\lambda+k\alpha_1}\varphi\otimes\bigwedge\nolimits_{\gamma_2+k,\lambda+k\alpha_1}\varphi\right)\right)\subset\bar{\theta}\left(\bigwedge\nolimits_{\geq\nu(t,\lambda+k\alpha_1),\lambda+k\alpha_1}\varphi\right)\subset\dot{M}_{t,\lambda},$$

and we have $E(2)\subset\dot{M}_{t,\lambda}$ by (3.2) and (3.3). □

4. Homology of ν-Schur complexes

Let $\varphi: G\to F$ and $Z=X\cup Y$ be as in the previous section. In this section, we assume that $m,n\geq 1$, $\varphi(y_1)=x_1$, and that $\varphi(G_1)\subset F_1$, where F_1 (resp. G_1) is the R-span of $X_1=\{x_2,\dots,x_m\}$ (resp. $Y_1=\{y_2,\dots,y_n\}$). Thus, $\varphi=\mathrm{id}_R\oplus\varphi_1$, where $\varphi_1: G_1\to F_1$ is the restriction of φ to G_1.

Definition 4.1. Let s, l, s' and l' be non-negative integers, λ/μ a relative row-sequence, and ν a row-sequence. We define

$$\tilde{B}^{s,l}_\nu(\lambda/\mu):=\{S\in\mathrm{Row}_{\geq\nu,\lambda/\mu}(Z,Y)\mid v_{\mathbb{N}}(S,\{x_1,y_1\})=s,\ v_{\{1,\dots,l\}}(S,\{y_1\})=0\}$$

and

$$\tilde{B}^{s,l,s',l'}_\nu(\lambda/\mu):=\{S\in\tilde{B}^{s,l}_\nu(\lambda/\mu)\mid v_{\{1,\dots,l\}}(S,\{x_1\})\geq s'\},$$

where

$$v_I(S,W)=\#\{(i,j)\in\Delta_{\lambda/\mu}\mid i\in I,\ S(i,j)\in W\}$$

for a set U, $I\subset\mathbb{N}$, $S\in\mathrm{Tab}_{\lambda/\mu}(U)$, and $W\subset U$. We define $\tilde{X}^{s,l}_\nu(\lambda/\mu)$ (resp. $\tilde{X}^{s,l,s',l'}_\nu(\lambda/\mu)$) to be the R-span of $\tilde{B}^{s,l}_\nu(\lambda/\mu)$ (resp. $\tilde{B}^{s,l,s',l'}_\nu(\lambda/\mu)$) in $\bigwedge_{\geq\nu,\lambda/\mu}\varphi$. When λ and μ are partitions, then the image $d_{\lambda/\mu}(\tilde{X}^{s,l}_\nu(\lambda/\mu))$ (resp. $d_{\lambda/\mu}(\tilde{X}^{s,l,s',l'}_\nu(\lambda/\mu))$) is denoted by $X^{s,l}_\nu(\lambda/\mu)$ (resp. $X^{s,l,s',l'}_\nu(\lambda/\mu)$).

Note that $\tilde{B}^{s,l}_\nu(\lambda/\mu)=\tilde{B}^{s,l,0,l'}$ so that $\tilde{X}^{s,l}_\nu(\lambda/\mu)=\tilde{X}^{s,l,0,l'}_\nu(\lambda/\mu)$ for any s, l and l'. It is easy to see that $\tilde{X}^{s,l,s',l'}_\nu(\lambda/\mu)$ is a free subcomplex of $\bigwedge_{\geq\nu,\lambda/\mu}\varphi$.

Note also that $\tilde{X}^{s,l,s',l'}_\nu(\lambda/\mu)$ and $X^{s,l,s',l'}_\nu(\lambda/\mu)$ are natural generalization of $\tilde{X}^{s,l,s',l'}_t(\lambda/\mu)$ and $X^{s,l,s',l'}_t(\lambda/\mu)$ [13, Section 2], respectively. The following lemmas are natural extensions of the results in [12, subsection 2.2] or [13, Section 4]. Almost all proofs are omitted, as they are obvious generalizations.

Lemma 4.2. *Let s, l, s' and l' be non-negative integers, λ/μ be a skew partition, and ν a row-sequence. Then we have:*

1. $L_{\nu,\lambda/\mu}\varphi=\bigoplus_{s\geq 0}X^{s,0}_\nu(\lambda/\mu)$.

2. *For each s, s′ and l′, we have a filtration*

$$X_\nu^{s,0,s',l'}(\lambda/\mu) \supset X_\nu^{s,1,s',l'}(\lambda/\mu) \supset \cdots \supset X_\nu^{s,l(\lambda/\mu),s',l'}(\lambda/\mu) \supset 0.$$

3. *For any s, l, s′ and l′, $X_\nu^{s,l,s',l'}(\lambda/\mu)$ is a free subcomplex of $L_{\nu,\lambda/\mu}\varphi$ with a free basis*

$$B_\nu^{s,l,s',l'}(\lambda/\mu) := d_{\lambda/\mu}(\mathrm{St}_{\lambda/\mu}(Z,Y) \cap \tilde{B}_\nu^{s,l,s',l'}(\lambda/\mu)).$$

It's underlying module is universally free on F_1 and G_1.

Lemma 4.3. *Let s, l, s′, l′, ν and λ/μ be as above. Then we have*
1. *If $s=0$, $\lambda_{l+1}=\lambda_{l+2}$ or $\lambda_{l+1}=\mu_{l+1}$, then $X_\nu^{s,l,s',l'}(\lambda/\mu)=X_\nu^{s,l+1,s',l'}(\lambda/\mu)$.*
2. *If $s>0$, $\lambda_{l+1}>\lambda_{l+2}$* and *$\lambda_{l+1}>\mu_{l+1}$, then we have the exact sequence*

$$0 \to X_\nu^{s,l+1,s',l'}(\lambda/\mu) \xrightarrow{\iota} X_\nu^{s,l,s',l'}(\lambda/\mu) \xrightarrow{v_\nu^{s,l}} X_\nu^{s-1,l,s',l'}((\lambda-\varepsilon_{l+1})/\mu)[-1] \to 0,$$

where ι is the inclusion map, and $v_\nu^{s,l}$ is the map induced by the map

$$\tilde{v}_\nu^{s,l}(\lambda/\mu): \tilde{X}_\nu^{s,l,s',l'}(\lambda/\mu) \to \tilde{X}_\nu^{s-1,l,s',l'}((\lambda-\varepsilon_{l+1})/\mu)[-1]$$

given by

$$\tilde{v}_\nu^{s,l}(\lambda/\mu)(S) = \begin{cases} (-1)^{\nu_{\{1,\ldots,l+1\}}(S,Y)}\bar{S} & (\text{if } S(\lambda+1,\lambda_{l+1})=y_1), \\ 0 & (\text{otherwise}) \end{cases}$$

for $S \in \tilde{B}_\nu^{s,l,s',l'}(\lambda/\mu)$ (the sign choice in the definition of $v_t^{s,l}$ in [12, p. 469] is erroneous). □

We denote $X_\nu^{s,l(\lambda/\mu),s',l'}(\lambda/\mu)$ by $X_\nu^{s,\infty,s',l'}(\lambda/\mu)$.

For a skew partition λ/μ, we set

$$\Gamma_s(\lambda/\mu) := \{\gamma : \text{partition} \mid \mu \subset \gamma \subset \lambda,\ \gamma/\mu\text{: a vertical } s\text{-strip}\},$$

and

$$\Gamma_s^{s',l'}(\lambda/\mu) := \left\{\gamma \in \Gamma_s(\lambda/\mu) \,\middle|\, \sum_{i=1}^{l'} (\gamma_i - \mu_i) \geq s'\right\}.$$

For $\gamma, \gamma' \in \Gamma_s(\lambda/\mu)$, we say that $\gamma \geq_{s',l'} \gamma'$ when
1. $\gamma \in \Gamma_s^{s',l'}(\lambda/\mu)$ and $\gamma' \notin \Gamma_s^{s',l'}(\lambda/\mu)$ or
2. Condition 1 does not hold, and $\gamma \geq \gamma'$.

The relation $\geq_{s',l'}$ is a total order, and is compatible with the order $\succeq$. Hence, when we set $M_\nu^{s',l'}(\gamma) = \sum_{\gamma' \geq_{s',l'} \gamma} \operatorname{Im} \Xi_{\gamma'}$, we have

$$X_\nu^{s,\infty,s',l'}(\lambda/\mu) = M_\nu^{s',l'}(\gamma_0),$$

where γ_0 is the smallest element of $\Gamma_s^{s',l'}(\lambda/\mu)$, and

$$\Xi_{\gamma'}: \textstyle\bigwedge_{\gamma'/\mu} R \otimes \bigwedge_{\nu,\lambda/\gamma'} \varphi_1 \to X_\nu^{s,\infty,s',l'}(\lambda/\mu)$$

is as in [12, p. 470, 13, Section 4].

Lemma 4.4. *$\{M_\nu^{s',l'}(\gamma)\}_{\gamma\in\Gamma_\nu^{s',l'}(\lambda/\mu)}$ is a filtration of $X_\nu^{s,\infty,s',l'}(\lambda/\mu)$ whose associated graded object is*

$$\bigoplus_{\gamma\in\Gamma_\nu^{s',l'}(\lambda/\mu)} L_{\gamma/\mu}R\otimes L_{\nu,\lambda/\gamma}\varphi_1.$$

Lemma 4.5. *Let λ/μ be a skew-partition with $l = l(\lambda/\mu) \geq 2$. Assume that $\lambda_l - \mu_l = 1$, and $s' = \tilde{\mu}_{\lambda_l}$. Then, for any $s \geq 0$ and a partition ν such that $\mu \subset \nu \subset \lambda$ and that $\nu_l = \mu_l$, the inclusion map*

$$X_\nu^{s,l-1,s',s+s'-l+1}(\lambda/\mu) \hookrightarrow X_\nu^{s,l-1}(\lambda/\mu)$$

is a quasi-isomorphism.

Proof. As in the proof of Lemma 4.3 of [13], the cokernel of the inclusion map agrees with the exact complex $L_{\lambda/\mu}\,\mathrm{id}_R\otimes X_\nu^{s+s'-l,\infty,s',s+s'-l}(\lambda/\gamma)$, where

$$\gamma = (\mu_1, \ldots, \mu_{s'}, \mu_{s'+1}+1, \ldots, \mu_l + 1). \quad \square$$

5. A vanishing theorem

In this section, we prove a vanishing theorem of the ν-Schur complex of the identity map. The theorem is a natural modification of [13, Theorem 4.4], but not a generalization. The idea of the proof of the theorem is the same as that of [13, Theorem 4.4], but we give a proof here for completeness.

In this section we consider $\mathrm{id}_F : F' \to F$, where F and F' are free R-modules of rank n with ordered bases $X = \{x_1 < \cdots < x_n\}$ and $X' = \{x'_1 > \cdots > x'_n\}$, respectively, and the map id_F is given by $\mathrm{id}_F(x'_i) = x_i$ for $1 \leq i \leq n$. We assume that $n \geq 1$ unless otherwise specified. With letting $G = F'$, $Y = X'$, $X_1 = \{x_2, \ldots, x_n\}$ and $Y_1 = \{x'_2, \ldots, x'_n\}$, we use the notation and terminology defined in the previous section for φ to our id_F freely. In particular, we consider $X < X'$ when we consider standardness of tableaux. We denote the R-spans of X_1 and Y_1 by F_1 and F'_1, respectively. The restriction of id_F on F'_1 is denoted by id_{F_1}.

Let λ/μ be a skew partition, and ν a row-sequence such that $\nu \preceq \lambda$. We set $\rho = \rho(\lambda/\mu, \nu)$, and we define $a(\lambda/\mu, \nu) = l(\lambda/\mu) - l(\rho/\mu) - 1$. When $\nu \npreceq \lambda$, we define $a(\lambda/\mu, \nu) = \infty$, as a convention.

Theorem 5.1. *Let λ/μ be a skew partition, and ν a row-sequence, and i, s, $l \geq 0$ with $l < l(\lambda/\mu)$. Then, we have $H_i(X_\nu^{s,l}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, \nu)$. In particular, we have $H_i(L_{\nu,\lambda/\mu}\,\mathrm{id}_F) = 0$ for $i \leq a(\lambda/\mu, \nu)$.*

Proof (*Double induction on $n = \mathrm{rank}\, F$ and $|\lambda/\mu|$*). Note that the first assertion in the theorem does not make sense when $n = 0$, but the second statement does, and it is obviously true (we use this in the induction argument when $n = 1$). We may assume that $\nu \preceq \lambda$ and $\nu = \rho(\lambda/\mu, \nu)$. Thus, ν is a partition such that $\mu \subset \nu \subset \lambda$. We may assume that $s > 0$, by induction assumption on n. We proceed by reverse induction on l. First, note that we may assume that $\nu_{l(\lambda/\mu)} = \mu_{l(\lambda/\mu)}$, or equivalently, $a(\lambda/\mu, \nu) \geq 0$.

Case a. First, consider the case $l = l(\lambda/\mu) - 1$. Note that we have $\nu \subset \lambda - \varepsilon_{l+1}$. By Lemma 4.3, we have an exact sequence

$$0 \to X_\nu^{s,\infty}(\lambda/\mu) \to X_\nu^{s,l}(\lambda/\mu) \to X_\nu^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)[-1] \to 0. \tag{5.1}$$

We consider the case $\lambda_{l+1} - \mu_{l+1} \geq 2$. In this case, we have $a((\lambda - \varepsilon_{l+1})/\mu, \nu) = a(\lambda/\mu, \nu)$. So, $H_i(X_\nu^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)[-1] = 0$ for $i \leq a(\lambda/\mu, \nu) + 1$ by induction assumption. On the other hand by Lemma 4.4, $X_\nu^{s,\infty}(\lambda/\mu)$ admits a filtration whose associated graded object is $\bigoplus_{\gamma \in \Gamma_s(\lambda/\mu)} L_{\nu, \lambda/\gamma}\,\mathrm{id}_F$. Since we have $a(\lambda/\mu, \nu) \leq a(\lambda/\gamma, \nu)$ for each gamma, we have $H_i(X_\nu^{s,\infty}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, \nu)$ by induction assumption. By the exact sequence (5.1), we have $H_i(X_\nu^{s,l}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, \nu)$, as desired.

Now we consider the case $\lambda_{l+1} - \mu_{l+1} \leq 1$. In this case, we have $\lambda_{l+1} - \mu_{l+1} = 1$, since $l + 1 = l(\lambda/\mu)$. Assume that $\mu_l \geq \lambda_{l+1}$. Then $\{(l + 1, \lambda_{l+1})\}$ is one of the connected components of $\Delta_{\lambda/\mu}$ (see [12, p. 471]), and in this case, we have

$$X_\nu^{s,l}(\lambda/\mu) \cong \mathrm{id}_{F_1} \otimes X_\nu^{s,\infty}((\lambda - \varepsilon_{l+1})/\mu) \bigoplus \mathrm{id}_R \otimes X_\nu^{s-1,\infty}((\lambda - \varepsilon_{l+1})/\mu).$$

Hence, $H_i(X_\nu^{s,l}(\lambda/\mu)) = 0$ for any $i \geq 0$ this case. So we may assume that $\mu_l = \mu_{l+1}$. By Lemma 4.5, $X_\nu^{s,l}(\lambda/\mu)$ is quasi-isomorphic to $X_\nu^{s,l,s',a}(\lambda/\mu)$, where $s' = \tilde{\mu}_{l+1}$ and $a = s + s' - l$. By Lemma 4.3, we have an exact sequence

$$0 \to X_\nu^{s,\infty,s',a}(\lambda/\mu) \to X_\nu^{s,l,s',a}(\lambda/\mu) \to X_\nu^{s-1,\infty,s',a}((\lambda - \varepsilon_{l+1})/\mu)[-1] \to 0. \tag{5.2}$$

For any $\gamma \in \Gamma_s^{s',a}(\lambda/\mu)$, we have $l(\lambda/\gamma) = l + 1$, as $(l + 1, \lambda_{l+1}) \in \Delta_{\lambda/\gamma}$. By induction assumption and Lemma 4.4 we have $H_i(X_\nu^{s,\infty,s',a}(\lambda/\mu) = 0$ for $i \leq a(\lambda/\mu, \nu)$. Moreover, we have $l((\lambda - \varepsilon_{l+1})/\gamma) = l$ for any $\gamma \in \Gamma_{s-1}^{s',a}((\lambda - \varepsilon_{l+1})/\mu)$, as $(l, \lambda_{l+1}) \in \Delta_{(\lambda - \varepsilon_{l+1})/\gamma}$. This shows that $H_i(X_\nu^{s-1,\infty,s',a}((\lambda - \varepsilon_{l+1})/\mu)[-1] = 0$ for $i \leq a(\lambda/\mu, \nu)$. By the exact sequence (5.2), we have $H_i(X_\nu^{s,l,s',a}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, \nu)$, as desired.

Case b. Now we consider the case $l < l(\lambda/\mu) - 1$. We may assume that $\lambda_{l+1} > \lambda_{l+2}, \mu_{l+1}$ by Lemma 4.3. By the same lemma, we have an exact sequence

$$0 \to X_\nu^{s,l+1}(\lambda/\mu) \to X_\nu^{s,l}(\lambda/\mu) \to X_\nu^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)[-1] \to 0.$$

So, by induction assumption on l, it suffices to show that $H_i(X_\nu^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)[-1]) = 0$ for $i \leq a(\lambda/\mu, \nu)$. By induction assumption on $|\lambda/\mu|$, it suffices to show that $a((\lambda - \varepsilon_{l+1})/\mu, \nu) \geq a(\lambda/\mu, \nu) - 1$. But this is trivial, as $l((\lambda - \varepsilon_{l+1})/\mu) = l(\lambda/\mu)$ and $l(\rho((\lambda - \varepsilon_{l+1})/\mu, \nu)/\mu) \leq l(\nu/\mu)$, unless $\nu \not\preceq \lambda - \varepsilon_{l+1}$. This completes the proof of the theorem. □

Corollary 5.2. *Let λ be a partition of length l, and $t \geq 1$. Then, we have $H_i(L_{\nu(t,\lambda),\tilde{\lambda}}\,\mathrm{id}_F) = 0$ for $i \leq 2l - 3$.*

6. The first syzygy

As a first application of Corollary 5.2, we give a new proof of Kurano's first syzygy theorem.

Let $\mathrm{id}_F : F' \to F$ be as in the previous section. As in Section 3, S denotes the polynomial ring $S(S_2F) \cong R[x_i \times x_j]_{1 \le i \le j \le n}$, and I_t denotes the ideal of S generated by all t-minors of the symmetric matrix $(x_i \times x_j)$.

Theorem 6.1 (Kurano [17, Corollary 5.5]). $\mathrm{Tor}_2^S(S/I_t, S/S_+)$ *is concentrated in degree* $t + 1$. *Or, equivalently, the relation module of* I_t *is generated by degree* $t + 1$ *elements.*

Proof. It suffices to prove that $T_j := [\mathrm{Tor}_2^S(S/I_t, S/S_+)]_j = 0$ for $j \ne t + 1$. Note that T_j is the homology group $H_1(\mathscr{I}^{t,j})$ of the R-free complex $\mathscr{I}^{t,j}$ by (3.1). As $H_0(\mathscr{I}^{t,j})$ is R-free for any j, we may assume that R is the ring of integers $\mathbb{Z}$ by universal coefficient theorem. Again by universal coefficient theorem, we may and shall assume that R is a field. It is clear that $T_j = 0$ for $j \le t$, as I_t is generated by degree t-elements.

As I_t is minimally generated by a Gröbner basis of I_t with respect to an appropriate monomial order [6], we have $T_j = 0$ for $j > 2t$ by Buchberger's theorem (see, e.g., [5,7]).

Thus, we may assume that $t + 2 \le j \le 2t$. In this case, by Lemma 3.7, there is a spectral sequence converging to $[\mathrm{Tor}_*^S(S/I_t, S/S_+)]$, whose E^1-term is $H_*(L_{\nu(t,\lambda),\tilde{\lambda}}\,\mathrm{id}_F)$ for various λ with $|\lambda| = j$. Thus, to prove that $T_j = 0$, it suffices to prove that $H_1(L_{\nu(t,\lambda),\tilde{\lambda}}\,\mathrm{id}_F) = 0$ for all partitions λ with $|\lambda| = j$. If $l(\lambda) \ge 2$, then this follows immediately from Corollary 5.2. If $\lambda = (j)$, then the complex $L_{\nu(t,\lambda),\tilde{\lambda}}\,\mathrm{id}_F$ is obtained by truncating the homotopically trivial complex $L_{\tilde{\lambda}}\,\mathrm{id}_F$ at degree $j - t$. Hence, we have $H_i(L_{\nu(t,\lambda),\tilde{\lambda}}\,\mathrm{id}_F) = 0$ for $i < j - t$. □

7. The second syzygy

In this section, the base ring R is assumed to be a field of characteristic p, unless otherwise specified. Let $\mathrm{id}_F : F' \to F$, $X = \{x_1 < \cdots < x_n\}$ and $X' = \{x'_1 > \cdots > x'_n\}$ be as in Section 5. $S = S(S_2F) \cong R[x_i \times x_j]_{1 \le i \le j \le n}$, and I_t are as in the previous section. We set the ith graded Betti number at degree j $\dim_R [\mathrm{Tor}_i^S(S/I_t, S/S_+)]_j$ of S/I_t by β_{ij}^R. The ith Betti number $\sum_j \beta_{ij}^R$ is denoted by β_i^R. The purpose of this section is to prove

Theorem 7.1. *If* $p = 3$, $n \ge 11$, *and* $t = 3$, *then we have* $\beta_{3,6}^R > 0$.

Proof. Thanks to the isomorphism (3.1), it suffices to show that $H_2(\mathscr{I}^{3,6}(\mathrm{id}_F)) \ne 0$ when $n = \mathrm{rank}\, F \ge 11$. Let us consider the filtration $\{M_{3,\lambda}\}_{|\lambda|=6}$ of $\mathscr{I}^{3,6}(\mathrm{id}_F)$. By

Lemma 3.7, the filtration induces a spectral sequence converging to $H_*(\mathscr{I}^{3,6}(\mathrm{id}_F))$ whose E^1 term is of the form

$$H_*(M_{3,\lambda}/\dot{M}_{3,\lambda}) = H_*(L_{\nu(3,\lambda),\tilde{\lambda}}\,\mathrm{id}_F).$$

The E^1-term $H_2(L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F)$, which corresponds to the partition $\lambda = (3,3)$ is isomorphic to the E^∞-term. We check this. The E^1-term $H_1(L_{\nu(3,\gamma),\tilde{\gamma}}\,\mathrm{id}_F) = 0$ for any partition γ such that $\gamma > \lambda$ and that $|\lambda| = 6$. This is a consequence of Corollary 5.2 when $l(\gamma) = 2$. When $\gamma = (6)$, then this is also clear, because we obtain $L_{(3,3),(6,6)}\,\mathrm{id}_F$ with truncating the homotopically trivial complex $L_{(6,6)}\,\mathrm{id}_F$ at degree 6 (so that $H_i(L_{(3,3),(6,6)}\,\mathrm{id}_F) = 0$ for $i < 6$). On the other hand, E^1- term $H_3(L_{\nu(3,\gamma),\tilde{\gamma}}\,\mathrm{id}_F) = 0$ for any partition γ such that $\gamma < \lambda$ and $|\lambda| = 6$. In fact, such a gamma has the length at least three, and we can invoke Corollary 5.2 again. Thus, the E^1-term $H_2(L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F)$ agrees with E^∞-term by a formal spectral sequence argument, using the facts above.

Hence, it suffices to show that $H_2(L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F) \neq 0$. The complex $L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F$ is a complex of $\mathrm{GL}(F)$-modules, and it decomposes into the direct sum

$$\bigoplus_{\rho \in \mathbb{N}_0^n} [L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F]_\rho,$$

where $[\ ]_\rho$ denotes the weight ρ-component (with respect to the basis $X = \{x_1, \dots, x_n\}$).

So it suffices to show that $H_2(C) \neq 0$, where C is the weight $(2, 1^{10})$-component

$$[L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F]_{(2,1^{10})}$$

of $L_{(3,3),(3,3,3,3)}\,\mathrm{id}_F$. We denote the weight $(2, 1^{10})$ by ω. We set $\tilde{C} = [\bigwedge_{\succeq(3,3),(3,3,3,3)}\mathrm{id}_F]_\omega$ so that $C = d_{(3,3,3,3)}(\tilde{C})$. We denote the set

$$\{S \in \mathrm{Row}_{\succeq(3,3),(3,3,3,3)}(X \cup X', X') \mid w(S) = \omega, \nu_{\mathbb{N}}(S, X') = r\}$$

by $\tilde{B}_r$, where

$$w(S) = (w_1(S), w_2(S), \dots, w_n(S))$$

is the vector given by $w_i(S) = \nu_{\mathbb{N}}(S, \{x_i, x_i'\})$. We also set

$$B_r = \tilde{B}_r \cap \mathrm{St}_{(3,3,3,3)}(X \cup X', X').$$

The set $\tilde{B}_r$ (resp. B_r) is a basis of $\tilde{C}_r$ (resp. C_r).

We define a linear form $\tilde{h}: \tilde{C}_2 \to R$ as follows. For an element $S \in \tilde{B}_2$, we define:

1. If $\nu_{\{1,2\}}(S, \{x_1\}) < 2$, then $\tilde{h}(S) = 0$.
2. If $\nu_{\{3\}}(S, X') = 2$, then $\tilde{h}(S) = 0$.
3. If $\nu_{\{4\}}(S, X') = 2$, then $\tilde{h}(S) = 0$.

4. If none of the above holds, then S is of the form

$$S = \begin{array}{lll} 1 & \sigma(2) & \sigma(3) \\ 1 & \sigma(4) & \sigma(5) \\ \sigma(6) & \sigma(7) & \boxed{\sigma(8)} \\ \sigma(9) & \sigma(10) & \boxed{\sigma(11)} \end{array}$$

for some (unique) $\sigma \in \mathrm{Aut}\{2, \dots, 11\} \cong \mathfrak{S}_{10}$, where in the expression of S, the number i denotes x_i, and framed $\boxed{i}$ denotes x'_i. We define $\tilde{h}(S) = (-1)^{\sigma} \in R$.

The linear form $\tilde{h}$ which satisfies 1–4 above exists uniquely.

We show that $\tilde{h}$ induces the linear form $h : C_2 \to R$. By Lemma 3.5 and [13, Lemma 7.5], the kernel of the map $d_{(3,3,3,3)} : \tilde{C}_2 \to C_2$ agrees with $\sum_{i=1}^{3} \sum_{j=1}^{3} E_{ij}$, where

$$E_{ij} = \square^{\rho(i,j)}_{(3,3,3,3)} \left(\bigwedge\nolimits_{(\rho(i,j)_1, \rho(i,j)_2), \rho(i,j)} \mathrm{id}_F \right)_{\omega}$$

for $i, j = 1, 2, 3$, here $\rho(i,j)$ denotes the row-sequence $(3,3,3,3) + j\alpha_i$. Let

$$T \in \bigcup_{i,j} \mathrm{Row}_{\geq (\rho(i,j)_1, \rho(i,j)_2), \rho(i,j)} (X \cup X', X')$$

be of weight ω (i.e., $v_{\mathbb{N}}(T, \{x_i, x'_i\}) = w_i$). The image of T by the box map can be expressed as

$$\square_{(3,3,3,3)}(T) = \sum_{m=1}^{r} c_m S_m \; (c_m = \pm 1,\ S_m \in \mathrm{Row}_{(3,3,3,3)}(X \cup X', X')).$$

If there is some m such that $\tilde{h}(S_m) \neq 0$, then by definition of $\tilde{h}$, T must be one of the following:

$$T_{1,1}(\sigma) = \begin{array}{llll} 1 & \sigma(2) & \sigma(3) & \sigma(4) \\ 1 & \sigma(5) & & \\ \sigma(6) & \sigma(7) & \boxed{\sigma(8)} & \\ \sigma(9) & \sigma(10) & \boxed{\sigma(11)} & \end{array} \qquad T_{1,2}(\sigma) = \begin{array}{lllll} 1 & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \\ 1 & & & & \\ \sigma(6) & \sigma(7) & \boxed{\sigma(8)} & & \\ \sigma(9) & \sigma(10) & \boxed{\sigma(11)} & & \end{array}$$

$$T_{2,1}(\sigma) = \begin{array}{llll} 1 & \sigma(2) & \sigma(3) & \\ 1 & \sigma(4) & \sigma(5) & \sigma(6) \\ \sigma(7) & \boxed{\sigma(8)} & & \\ \sigma(9) & \sigma(10) & \boxed{\sigma(11)} & \end{array} \qquad T_{2,2}(\sigma) = \begin{array}{lllll} 1 & \sigma(2) & \sigma(3) & & \\ 1 & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) \\ \boxed{\sigma(8)} & & & & \\ \sigma(9) & \sigma(10) & \boxed{\sigma(11)} & & \end{array}$$

$$T_{3,1}(\sigma) = \begin{array}{llll} 1 & \sigma(2) & \sigma(3) & \\ 1 & \sigma(4) & \sigma(5) & \\ \sigma(6) & \sigma(7) & \sigma(8) & \boxed{\sigma(9)} \\ \sigma(10) & \boxed{\sigma(11)} & & \end{array} \qquad T_{3,2}(\sigma) = \begin{array}{lllll} 1 & \sigma(2) & \sigma(3) & & \\ 1 & \sigma(4) & \sigma(5) & & \\ \sigma(6) & \sigma(7) & \sigma(8) & \sigma(9) & \boxed{\sigma(10)} \\ \boxed{\sigma(11)} & & & & \end{array}$$

$$U_{3,2}(\sigma) = \begin{array}{llll} 1 & \sigma(2) & \sigma(3) & \\ 1 & \sigma(4) & \sigma(5) & \\ \sigma(6) & \sigma(7) & \sigma(8) & \boxed{\sigma(9)\quad \sigma(10)} \\ \sigma(11) & & & \end{array} \qquad U_{3,3}(\sigma) = \begin{array}{lllll} 1 & \sigma(2) & \sigma(3) & & \\ 1 & \sigma(4) & \sigma(5) & & \\ \sigma(6) & \sigma(7) & \sigma(8) & \sigma(9) & \boxed{\sigma(10)\quad \sigma(11)} \end{array},$$

where $\sigma \in \operatorname{Aut}\{2, \ldots, 11\}$. As the characteristic of R is three, a straightforward computation will show that the tableaux above are mapped to zero by $\tilde{h} \circ \square_{(3,3,3,3)}$. Thus, $\tilde{h}$ induces $h: C_2 \to R$.

Next, we show that h induces a map $H_2(C) \to R$. As $H_2(C) = Z_2(C)/B_2(C)$, where $Z_2(C) = \operatorname{Ker} \partial \cap C_2$ and $B_2(C) = \operatorname{Im} \partial \cap C_2$, it suffices to show that $h \circ \partial$ vanishes on C_3. To verify this, we take a tableaux $T \in B_3$, and prove $\tilde{h}(\partial T) = 0$. By definition of $\tilde{h}$ (the condition 3), we may assume that $v_{\{4\}}(T, X') = 2$. If $v_{\mathbb{N}}(T, \{x_1\}) = 0$, then ∂T is a linear combination of tableaux S_m such that $v_{\mathbb{N}}(S, \{x_1\}) \le 1$, and it is clear that $\tilde{h}(\partial T) = 0$. So we may assume that $v_{\mathbb{N}}(T, \{x_1\}) \ge 1$. First we consider the case $v_{\mathbb{N}}(T, \{x_1\}) = 2$. Then, as T is standard of weight ω, T looks like

$$T = \begin{array}{lll} 1 & \sigma(2) & \sigma(3) \\ 1 & \sigma(4) & \sigma(5) \\ \sigma(6) & \sigma(7) & \boxed{\sigma(8)} \\ \sigma(9) & \boxed{\sigma(10)} & \boxed{\sigma(11)} \end{array} \quad \text{or} \quad T = \begin{array}{lll} 1 & \sigma(2) & \sigma(3) \\ 1 & \sigma(4) & \sigma(5) \\ \sigma(6) & \sigma(7) & \sigma(8) \\ \boxed{\sigma(9)} & \boxed{\sigma(10)} & \boxed{\sigma(11)} \end{array}$$

for some $\sigma \in \operatorname{Aut}\{2,, \ldots, 11\}$. A straightforward computation will show that this is annihilated by $\tilde{h} \circ \partial$.

Next, consider the case $v_{\mathbb{N}}(T, \{x_1\}) = 1$. Then, T is of the form

$$T = \begin{array}{lll} 1 & \sigma(2) & \sigma(3) \\ \sigma(4) & \sigma(5) & \sigma(6) \\ \sigma(7) & \sigma(8) & \boxed{\sigma(9)} \\ \sigma(10) & \boxed{\sigma(11)} & \boxed{1} \end{array}$$

for some $\sigma \in \operatorname{Aut}\{2, \ldots, 11\}$. Again by a straightforward computation (using that the characteristic is three), we see that $\tilde{h}(\partial(T)) = 0$ in R. So h induces $\bar{h}: H_2(C) \to R$.

To prove that $H_2(C) \neq 0$, it suffices to show that $\bar{h} \neq 0$. We consider the element

$$A = \partial \left(\begin{array}{lll} 1 & 2 & 3 \\ 1 & 4 & \boxed{5} \\ 6 & 7 & \boxed{8} \\ 9 & 10 & \boxed{11} \end{array} \right) \in C_2.$$

It is obvious that $A \in Z_2(C)$. On the other hand, a straightforward computation will show that $h(A) = 1$. Hence, we have $\bar{h}(\bar{A}) \neq 0$, where $\bar{A}$ denotes the class of A in $H_2(\mathrm{C})$. $\square$

Remark 7.2. It is known that $\beta^{\mathbb{Q}}_{3,j} = 0$ unless $j = t + 2$ [16]. Hence, by the theorem, we have $\beta^{\mathbb{F}_3}_{3,6} > \beta^{\mathbb{Q}}_{3,6} = 0$ when $t = 3$ and $n \geq 11$. So the third Betti number β_3 of S/I_t depends on characteristic of the base field in this case.

Corollary 7.3 (cf. Andersen [2, Corollary 5.4.2]). *Let us consider the base ring $R = \mathbb{Z}$, the ring of integers. If $n \geq 11$, then there is no minimal free resolution of S/I_3, where we say that a finite free S-complex $\mathbb{F}$ is minimal when the boundary maps of $\mathbb{F} \otimes_S S/S_+$ are zero maps.*

Proof. The existence of such an $\mathbb{F}$ would imply that the all Betti numbers of S/I_t were independent of the field, because $k \otimes_{\mathbb{Z}} \mathbb{F}$ would be the minimal free resolution of $k \otimes_{\mathbb{Z}} S/I_t$ for any filed k. □

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